

How classical is the quantum universe?

Maurice de Gosson*

Current address: *Universität Wien*
Fakultät für Mathematik, NuHAG
Nordbergstrasse 15, AT-1090 Wien
maurice.degosson@gmail.com

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Abstract

We discuss two topics that are usually considered to be exclusively "quantum": the Schrödinger equation, and the uncertainty principle. We show (or rather recall) that the Schrödinger equation can be derived from Hamilton's equations using the metaplectic representation. We also show that the uncertainty principle, stated in the form of the Robertson-Schrödinger-Heisenberg inequalities can be formulated in perfectly classical terms using the topological notion of symplectic capacity.

To my parents

1 Introduction

In his recent contribution [17] to the conference *Everett at 50* Hartle observes that

"...The most striking observable feature of our indeterministic quantum universe is the wide range of time, place, and scale on which the deterministic laws of classical physics hold to an excellent approximation."

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In this short essay I will try to complement Hartle’s discussion by exposing two *mathematical* facts that are usually ignored by physicists and, which seem to suggest that the quantum universe is, in a sense, far more classical than it seems at first sight. The first of these facts is about the Schrödinger equation; it should be known by all physicists but experience shows that this is not the case¹. The second is about the uncertainty principle; recent advances in symplectic topology of a very subtle nature, suggest that there exists an uncertainty principle in classical mechanics which is formally absolutely similar to a refined version of the Heisenberg inequalities.

More precisely, I want to point out that:

- The Schrödinger equation can be autonomously be derived from Hamilton’s equations of motion; by “autonomously” I mean without recourse to any extraneous ad hoc physical assumption. This possibility has been known for a long time by mathematicians working in representation theory, and is an immediate consequence of a property of the metaplectic representation of the symplectic group. The punchline is that Schrödinger’s equation is equivalent to Hamilton’s equations of motion!
- The uncertainty principle of quantum mechanics is already present, at least at a formal level, in classical mechanics in its Hamiltonian formulation. This is a consequence of a difficult result from symplectic topology, known as Gromov’s non-squeezing theorem. That theorem, which is a considerable refinement of Liouville’s theorem on conservation of phase-space volume under canonical transformations, was only proved in 1985, and is therefore not widely known by mathematicians –let alone physicists. Its consequences have certainly not been fully exploited yet.

2 Metaplectic Group and Schrödinger Equation

2.1 The case of quadratic Hamiltonians

Consider a system consisting of N particles moving in physical 3-dimensional space. We assume that the phase-space evolution of that system is governed

¹A few years ago, while being a visiting professor at Yale, I was invited by the mathematical physics group of a famous University located in new Jersey to give a talk on the topic. My claim almost triggered a riot among my colleagues physicists!

by Hamilton's equations

$$\frac{dx_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial x_j} \quad (1)$$

where x_1, x_2, x_3 (resp. p_1, p_2, p_3) are the position (resp. momentum) coordinates of the first particle, and so on. (A standard reference for the Hamiltonian mechanics I will be using can be found in any of the editions or re-editions of Goldstein [7]). We now make the following assumption on the Hamilton function H : it is a homogeneous quadratic polynomial in the variables x_j, p_k . In this case the flow determined by the system of differential (1) is linear, and consists of linear canonical transformations, that is of *symplectic matrices*. Setting $x = (x_1, \dots, x_{3N})^T$, $p = (p_1, \dots, p_{3N})^T$ this means that the solution of (1) is given, at time t , by

$$\begin{pmatrix} x(t) \\ p(t) \end{pmatrix} = S_t \begin{pmatrix} x(0) \\ p(0) \end{pmatrix}$$

where S_t is a $6N \times 6N$ real matrix such that

$$S_t J S_t^T = S_t^T J S_t = J$$

where J is the standard symplectic matrix: $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ where 0 and I are, respectively, the zero and unity $3N \times 3N$ matrices. In fact, if one writes the Hamiltonian H in the form

$$H(x, p) = \frac{1}{2} z^T M z, \quad z = \begin{pmatrix} x \\ p \end{pmatrix} \quad (2)$$

where M is symmetric, then the matrices S_t are explicitly given by the exponential

$$S_t = e^{tJM}.$$

Now, the set of all matrices S obeying the relations $SJS^T = S^TJS = J$ form a group, namely the well-known symplectic group; we denote it by $\text{Sp}(6N)$. Thus, when t varies the matrices S_t will describe a curve Σ in the symplectic group, and that curve passes passing the identity at time $t = 0$. A fundamental fact is now that $\text{Sp}(6N)$ has a double covering group. This covering is called the metaplectic group; we denote it by $\text{Mp}(6N)$. It is unique, as an abstract group. However –and this is crucial for the rest of the discussion– it can be realized, in infinitely many ways, as a group of unitary operators acting on $L^2(\mathbb{R}^{3N})$ (the square-integrable functions defined

on configuration space \mathbb{R}^{3N}); these groups are parametrized by a positive parameter and the copy $\text{Mp}^\varepsilon(6N)$ corresponding to the choice ε for this parameter will contain the Fourier-like transform \widehat{F}^ε defined by

$$\widehat{F}\psi(p) = \left(\frac{1}{2\pi i\varepsilon}\right)^{3N/2} \int_{\mathbb{R}^{3N}} e^{\frac{i}{\varepsilon}p \cdot x} \psi(x) dx. \quad (3)$$

Let us now fix once for all ε . Using a standard property from the theory of covering groups (the “path lifting property”), one proves that the curve Σ induces unambiguously a unique curve $\widehat{\Sigma}$ in $\text{Mp}^\varepsilon(6N)$ passing through the identity operator at time $t = 0$; it is the unique curve having this property and such that the projection of a point \widehat{S}_t of $\widehat{\Sigma}$ “down to $\text{Sp}(6N)$ ” is precisely the symplectic matrix S_t . In view of what has been said before, the curve $\widehat{\Sigma}$ consists of unitary operators acting on $L^2(\mathbb{R}^{3N})$; let us let \widehat{S}_t act on a smooth square-integrable function ψ_0 ; this defines a function of both position x and time t :

$$\psi(x, t) = \widehat{S}_t \psi_0(x). \quad (4)$$

The notion is intended to suggest that $\psi(x, t)$ might be some kind of “wave-function”. This is indeed the case: one proves (see below for a sketch of the proof) that $\psi(x, t)$ satisfies the Schrödinger-like equation

$$i\varepsilon \frac{\partial}{\partial t} \psi(x, t) = H(x, -i\varepsilon \nabla_x) \psi(x, t) \quad (5)$$

where $H(x, -i\varepsilon \nabla_x)$ is the partial differential operator obtained from the Hamilton function H through the symmetrized “quantization rules” $x_j \longrightarrow \widehat{x}_j$ (multiplication by x_j), $p_j \longrightarrow \widehat{p}_j = -i\varepsilon \partial / \partial x_j$, and $x_j p_k \longrightarrow \frac{1}{2}(\widehat{x}_j \widehat{p}_k + \widehat{p}_k \widehat{x}_j)$. Thus, if we choose the value of the arbitrary parameter ε to be $\hbar = h/2\pi$, then we obtain exactly Schrödinger’s equation

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = H(x, -i\hbar \nabla_x) \psi(x, t). \quad (6)$$

Admittedly, a mathematical equation is not a physical theory; to give the equations above a physical sense, one needs an interpretational apparatus, provided by physical considerations. Still, the equation is there; our argument –which is solely based on mathematical arguments, and does not invoke any physical assumption– shows in the end that equation (5), and hence also Schrödinger’s equation (6) are mathematically equivalent to Hamilton’s equations!

This observation is actually closely related to the fact that Ehrenfest’s equation

$$m \frac{d^2 \langle x \rangle}{dt^2} = - \left\langle \frac{\partial V}{\partial x}(x) \right\rangle \quad (7)$$

discussed by Hartle becomes

$$m \frac{d^2 \langle x \rangle}{dt^2} = - \frac{\partial V}{\partial x}(\langle x \rangle) \quad (8)$$

when the potential V is quadratic; in fact one can prove that for all quadratic Hamiltonians the time-evolution of the averages $\langle x_j \rangle$ and $\langle p_j \rangle$ is governed by the Hamilton equations (1) (formula (8) is an immediate consequence of this statement when H is of the type “kinetic energy plus potential V ”).

2.2 The Feynman integral

Here are two very simple explicit examples; we work in spatial dimension 1 for notational simplicity, but everything carries trivially through in higher dimensions. Assume first that $H = p^2/2m$, the free particle Hamiltonian. Then

$$S_t = \begin{pmatrix} 1 & t/m \\ 0 & 1 \end{pmatrix}$$

and, using general formulae for the metaplectic representation, one finds that the solution $\psi(x, t) = \widehat{S}_t \psi_0(x)$ of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) \quad (9)$$

with initial datum ψ_0 is given, for $t \neq 0$, by

$$\psi(x, t) = \int_{-\infty}^{\infty} K_t(x, y) \psi_0(y) dy \quad (10)$$

where the kernel function is given by

$$K_t(x, y) = (e^{i\pi/4})^{\text{sign } t} \sqrt{\frac{m}{2\pi\hbar|t|}} \exp \left[\frac{i}{\hbar} \frac{m(x-y)^2}{2t} \right]. \quad (11)$$

Suppose next that H is the harmonic oscillator Hamiltonian; for simplicity we choose $m = \omega = 1$ so that $H = \frac{1}{2}(p^2 + x^2)$; in this case the solution of Schrödinger's equation

$$i\hbar \frac{\partial}{\partial t} \psi(x, t) = \frac{1}{2} \left(-\hbar^2 \frac{\partial^2}{\partial x^2} + x^2 \right) \psi(x, t) \quad (12)$$

is given by (10), where the kernel is this time (for $t \neq n\pi$)

$$K_t(x, y) = i^{-[t/\pi]} \sqrt{\frac{1}{2\pi\hbar|\sin t|}} \exp \left[\frac{i}{2\hbar} \frac{(x^2 + y^2) \cos t - xy}{\sin t} \right]. \quad (13)$$

These formulae are of course well known by quantum physicists; they can be found for instance in Feynman–Hibbs [6] (but beware of misprints!) where they are presented as cases where the “Feynman path integrals” can “be done exactly”. The rub is that the Feynman integral approach is as string theory: it is not even wrong! A Feynman integral is an object which does not (outside a few cases) make sense mathematically, and hence does not exist². What happens is that the quadratic nature of the Hamiltonians lead to correct expressions for the “actions” via the Hamilton–Jacobi equation; this amounts to find exact generating functions for the Hamiltonian (“Hamilton’s two-point characteristic function”). It is thus clear if one looks at the derivation of formulae (11) and (13) in texts using the Feynman approach (for instance Schulman [28]) that these methods only reconstruct the metaplectic representation via a highly illegitimate legerdemain!

2.3 How far further can we go?

The reader will of course object that our considerations apply *strictu sensu* only to a very small class of physical systems, which are variations on the theme of the free particle, or the harmonic oscillator. I totally agree with this objection, especially since it is well-known that the quantum behavior of systems with quadratic Hamiltonians is very close to the classical behavior³, as is exemplified by Ehrenfest’s theorem which reduces to the classical Hamilton equations in this case. There is, moreover, a famous mathematical theorem, due Groenewold and Hove that says that we cannot use the metaplectic representation to construct solutions to Schrödinger’s equations for general Hamiltonians (the proof is actually rather complicated; the mathematically minded reader can have a look at the proof in Chapter 1 of Guillemin and Sternberg [16]). But this “no-go” result does not of course mean that there is no way to derive the Schrödinger equation from Hamilton’s equations of motion.

The first step towards such a program is easy, it is actually just a rather straightforward extension of the quadratic case. Assume that H is a non-homogeneous polynomial of degree two in the position and momentum variables. With the notation of (2) we can write

$$H(x, p) = \frac{1}{2} z^T M z + u^T z$$

²I admit that I am being a little bit unfair and grouchy at this stage; Feynman integrals certainly have a good heuristic value in many cases.

³One of the best studies of that kind of “classical vs. quantum” situation is –for my money– Littlejohn’s seminal paper [20].

where u is some given vector. The flow determined by the corresponding Hamilton equations no longer consists of symplectic matrices, but rather of affine canonical transformations (that is of linear symplectic transformations followed (or preceded) by a phase-space translation). Such transformations again form a group, the inhomogeneous symplectic group $\text{ISp}(3N)$ (it is the semi-direct product of the symplectic group and of the translation group). It turns out that we can repeat the same procedure as in the linear case, and show that there is, for every $\varepsilon > 0$, a one-to-one correspondence between continuous curves in $\text{ISp}(3N)$, and curves in a group of unitary operators, denoted by $\text{IMp}^\varepsilon(3N)$ and called the *inhomogeneous metaplectic group*. $\text{IMp}^\varepsilon(3N)$ consists of operators in $\text{Mp}^\varepsilon(3N)$ composed (on the left, or the right) with the Heisenberg operators

$$\widehat{T}(x_0, p_0)\psi(x) = \exp\left[\frac{i}{\varepsilon}(p_0 \cdot x - \frac{1}{2}p_0 \cdot x_0)\right]\psi(x - x_0)$$

familiar from the Schrödinger representation of the Heisenberg group when $\varepsilon = \hbar$ (the dot \cdot stands for the usual scalar product of vectors). Everything now carries over mutatis mutandis, and we conclude that, again, Hamilton's equations are mathematically equivalent to the Schrödinger-type equation associated with the non-homogeneous Hamiltonian H .

Can we do anything similar in more general cases? Yes, we can. Without going too much into details (our approach is here really sketchy) the procedure works as follows. Let f_t be the flow determined by the Hamilton equations (1). We now use the following trick, called the “nearby orbit method” (see Littlejohn [20] for a review of the method). It consists in replacing the Hamiltonian function H by its Taylor series to the second order around a point $z_t = f_t(z_0)$ where the initial point $z_0 = (x_0, p_0)$ is arbitrary. One thus replaces H by the always time-dependent Hamiltonian

$$H_{z_0}(z, t) = H(z_t) + \nabla_z H(z_t) \cdot (z - z_t) + \frac{1}{2}H''(z_t)(z - z_t) \cdot (z - z_t)$$

($H''(z_t)$ is the Hessian matrix of H calculated at z_t). Since $H_{z_0}(z, t)$ is a second degree polynomial in the position and momentum variables, our discussion above applies, and the Hamilton equations for H_{z_0} define a flow $f_{z_0, t}$ consisting of affine symplectic transformations, i.e. each $f_{z_0, t}$ is in the inhomogeneous group $\text{ISp}(3N)$. Now we make the following observation: when t varies $f_{z_0, t}(z_0)$ is just z_t , the solution of Hamilton's equations with initial datum t . Expressed in geometric terms this means that every Hamiltonian trajectory comes from an affine flow (but this flow depends each time on the initial point). This fact is well-known, and has been used with profit to

construct short-time solutions for Schrödinger’s equation with initial datum a narrow wavepacket, by propagating the center of this wavepacket along the classical curve (see again Littlejohn [20]; also de Gosson [12]): for this purpose, it suffices to lift, as we did before, the affine Hamiltonian flow to the inhomogeneous metaplectic group. Using the theory of Gabor frames from time-frequency analysis it is then possible to write such short-time solutions for arbitrary wavepackets; the validity of these solutions breaks down after some time (called “Ehrenfest time” in the literature), however there asymptotic validity for short times is sufficient to construct exact solutions by a Lie-Trotter argument. So (up to mathematical difficulties we do not discuss here) one ends up with wavepackets obeying Schrödinger’s equation.

3 Uncertainty Principle and Symplectic Camel

3.1 Gromov’s non-squeezing theorem

Hamiltonian motions consist of canonical transformations, and are thus volume preserving: this is Liouville’s theorem, one of the best known results from elementary statistical mechanics. Liouville’s theorem is perhaps also one of the most *understated* because in addition of being volume-preserving Hamiltonian motions have a surprising –I am tempted to say an *extraordinary*– additional property as soon as the number of degrees of freedom is superior to one. Let me shortly describe this property. Assume that we are dealing with a Hamiltonian system consisting of a large number N of particles. If the points are sufficiently close to each other and in sufficiently large number we may, with a good approximation, identify that population with a “cloud” of phase space fluid. Suppose that this cloud is, at time $t = 0$ spherical so we identify it with a ball

$$B(r) : |x|^2 + |p|^2 \leq r^2$$

where $|x|^2 = x_1^2 + \cdots + x_n^2$ and $|p|^2 = p_1^2 + \cdots + p_n^2$. The orthogonal projection of that ball on any plane of coordinates x_j, p_k will obviously be a circle with area πr^2 . From now on we assume that this plane is a plane of *conjugate coordinates*, that is x_1, p_1 or x_2, p_2 , etc. Let us watch the motion of this spheric phase-space cloud as time evolves. It will distort and may take after a while a very different shape, while keeping constant volume in view of Liouville’s theorem. However –and this is the surprising result– the projections of that deformed ball on any of the planes of conjugate coordinates will never decrease below its original value πr^2 ! If we had

chosen, on contrary, a plane of non-conjugate coordinates (such as x_1, p_2 or x_1, x_2 , for example) then there would be no obstruction for the projection to become arbitrarily small. This fact, which is mathematical theorem proved in 1985 by Gromov [15], is of course strongly reminiscent of the uncertainty principle of *quantum* mechanics, of which it can be viewed as a *classical* version!

This is in effect an extraordinary result, because it seems at first sight to conflict with the usual conception of Liouville's theorem: according to conventional wisdom, the ball $B(r)$ can be stretched in all directions, and eventually get very thinly spread out over huge regions of phase space, so that the projections on any plane could *a priori* become arbitrary small after some (perhaps very long) time t . In fact, one may very well envisage that the larger the number N of degrees of freedom, the more that spreading will have chances to occur since there are more and more directions in which the ball is likely to spread! This possibility has led to many philosophical speculations about the stability of Hamiltonian systems. For instance, in his 1989 book Roger Penrose ([24], p.174–184) comes to the conclusion that phase space spreading suggests that '*classical mechanics cannot actually be true of our world*' (p. 183, l.3). He however adds that "*quantum effects can prevent this spreading*" (p. 184, l. 9). Penrose's second observation goes right to the point: while phase space spreading a priori opens the door to classical chaos, quantum effects have a tendency to 'tame' the behavior of physical systems by blocking and excluding most of the classically allowed motions. However, the phenomena we shortly described above show that there is a similar taming in Hamiltonian mechanics preventing anarchic and chaotic spreading of the ball in phase space which would be possible if it were possible to stretch it inside arbitrarily thin tubes in directions orthogonal to the conjugate planes.

Now, why do we refer to a *symplectic camel* in the title of this section? This is because one can restate Gromov's theorem in the following way: there is no way to deform a phase space ball using canonical transformations in such a way that we can make it pass through a hole in a plane of conjugate coordinates x_j, p_j if the area of that hole is smaller than that of the cross-section of that ball. Recalling now that in *Matthew* 19(24) it is stated that

‘...Then Jesus said to his disciples, ‘Amen, I say to you, it will be hard for one who is rich to enter the kingdom of heaven. Again I say to you, it is easier for a camel to pass through the eye of a needle than for one who is rich to enter the kingdom of God’.⁴

⁴Also see St Luke 18(25) and Mk 10(25).

we see that in this case the Biblical camel is symplectic! For this reason we will refer to Gromov's theorem and its variant just described as the *principle of the symplectic camel*.

Our discussion above was of a purely qualitative nature. It turns out that we can do better, and produce quantitative statements using the principle of the symplectic camel. For this purpose it is very useful to introduce the topological notion of *symplectic capacity*.

3.2 Symplectic capacities

Consider an arbitrary region Ω in phase space \mathbb{R}^{6N} ; this region may be large, or small, bounded, or unbounded. By definition the Gromov capacity of Ω is the (possibly infinite) number $c_{\min}(\Omega)$ which is calculated as follows: let again $B(r)$ be a phase space ball with radius r , and assume first that there exists no canonical transformation sending that ball inside Ω . We will then say that $c_{\min}(\Omega) = 0$. Assume next that there are canonical transformations sending $B(r)$ in Ω . The supremum R of all the radii r for which this is possible is called the *symplectic radius* of Ω and we define the Gromov capacity of Ω by the formula $c_{\min}(\Omega) = \pi R^2$. Thus $c_{\min}(\Omega) = \pi R^2$ means that one can find canonical transformations sending $B(r)$ inside Ω for all $r < R$, but that no canonical transformation will send a ball with radius larger R inside that set. By its very definition we see that the Gromov capacity is a symplectic invariant, that is

$$c_{\min}(f(\Omega)) = c_{\min}(\Omega) \text{ if } f \text{ is canonical;} \quad (14)$$

it is obviously also monotone:

$$c_{\min}(\Omega) \leq c_{\min}(\Omega') \text{ if } \Omega \text{ is a subset of } \Omega' \quad (15)$$

and 2-homogeneous under phase space dilations:

$$c_{\min}(\lambda\Omega) = \lambda^2 c_{\min}(\Omega) \quad (16)$$

(λ a scalar; $\lambda\Omega$ consists of all points $(\lambda\Omega x, \lambda\Omega p)$ such that (x, p) is in Ω). However, the most striking property is the following: let us denote by $Z_j(R)$ the phase-space cylinder based on the plane of conjugate variables: it consists of all phase space points whose j -th position and momentum coordinate satisfy $x_j^2 + p_j^2 \leq R^2$. We have

$$c_{\min}(B(R)) = \pi R^2 = c_{\min}(Z_j(R)). \quad (17)$$

While the equality $c_{\min}(B(R)) = \pi R^2$ is immediate by definition of c_{\min} , the equality $c_{\min}(Z_j(R)) = \pi R^2$ is just a reformulation of Gromov's non-squeezing theorem, and hence very a deep property! In fact Gromov's theorem says that there is no way we can squeeze a ball with radius $R' > R$ inside that cylinder, because if we could then the orthogonal projection of the squeezed ball would be greater than the cross-section πR^2 of the cylinder, contradicting Gromov's theorem. Thus $c_{\min}(Z_j(R)) \leq \pi R^2$. That we actually have equality is immediate, since we can translate the ball $B(R)$ inside $Z_j(R)$ and phase space translations are canonical transformations in their own right.

More generally one calls symplectic capacity any function associating to subsets Ω of phase space a non-negative number $c(\Omega)$, or $+\infty$, and for which the properties (14), (15), (16), and (17) are verified (see Hofer and Zehnder's book [18] for very interesting examples. A caveat: the reading of this book requires some expertise in topology and geometry; in [8, 11] I have given a "milder" –but also far less complete– treatment of this topic). There exist infinitely many symplectic capacities⁵, and the Gromov capacity is the smallest of all: $c_{\min}(\Omega) \leq c(\Omega)$ for all Ω and c . Is there a "biggest" symplectic capacity c_{\max} ? Yes there is, and it is actually constructed by using again Gromov's non-squeezing theorem. It is defined as follows: suppose that no matter how large we choose r there exists no canonical transformation sending Ω inside a cylinder $Z_j(r)$. We then write any $c_{\max}(\Omega) = +\infty$. Suppose that, on the contrary, there are canonical transformations sending Ω inside some $Z_j(r)$. and let R be the supremum of all such r . Then we set $c_{\max}(\Omega) = \pi R^2$. Using the definition of a symplectic capacity it is not difficult to show that c_{\max} is a symplectic capacity, and that we have

$$c_{\min}(\Omega) \leq c(\Omega) \leq c_{\max}(\Omega) \quad (18)$$

for every symplectic capacity c . Now, having in mind the uncertainty principle, a very nice fact is that all symplectic capacities agree on phase space ellipsoids, and can be calculated as follows: assume that Ω is the ellipsoid centered at z_0 and given by the condition

$$(z - z_0)^T M (z - z_0) \leq 1 \quad (19)$$

where M is some positive-definite matrix. Consider now the eigenvalues of the product matrix JM ; they are the same as those of the antisymmetric matrix $M^{1/2}JM^{1/2}$ and are hence pure imaginary, that is of the type

⁵It seems to have become a kind of sport in symplectic topology to invent new capacities!

$\pm i\lambda_1, \dots, \pm i\lambda_{3N}$ where $\lambda_j > 0$. We have

$$c(\Omega) = \frac{\pi}{\lambda_{\max}} \quad (20)$$

where λ_{\max} is the largest of all the positive numbers λ_j

We will make use in next subsection of a weaker form of symplectic capacity, the so-called linear symplectic capacity c_{lin} (it should actually be rather called an “affine” capacity, but we are complying with usage). It is defined exactly as the Gromov capacity c_{min} but one restricts oneself to the use of only affine symplectic transformations: $c_{\text{lin}}(\Omega) = \pi R^2$ is thus the supremum of all numbers πr^2 such that there exists an affine transformation in $\text{ISp}(3N)$ sending the ball $B(r)$ inside Ω . The properties of c_{lin} are quite similar to those of an arbitrary symplectic capacity, except that we must replace the symplectic invariance formula (14) by the weaker condition

$$c_{\text{lin}}(f(\Omega)) = c_{\text{lin}}(\Omega) \text{ if } f \text{ is in } \text{ISp}(3N).$$

It turns out that if Ω is an ellipsoid then $c_{\text{lin}}(\Omega)$ is again given by formula (20); thus all symplectic capacities, including the linear one agree on ellipsoids.

3.3 The Heisenberg uncertainty principle

Assume that we have a cloud of phase space points concentrated in some bounded region Ω ; we do not assume that this cloud is spherical as we did when we discussed Gromov’s theorem, but just that it is a convex set. This is of course always feasible by choosing for Ω the convex hull of the cloud, that is, the intersection of all convex sets containing it. To make things “look quantum” we introduce Planck’s constant $h = \pi\hbar$ and assume that the linear symplectic capacity of Ω is $c_{\text{lin}}(\Omega) \geq \frac{1}{2}h$ (this is a pedagogical trick intended to create a surprise effect; we could have used as well ε instead of \hbar as in the first part of this paper: \hbar and h are just positive parameters having a priori no physical meaning). We now make the following remark: the convexity of Ω implies that there exists a unique ellipsoid \mathcal{J}_Ω contained in Ω and having maximal volume among all other ellipsoids contained in Ω . It is called the John ellipsoid; its existence was proven by Fritz John [19] in 1948 (see the nice paper [2] by Ball for an extension of that result). I claim that we have $c_{\text{lin}}(\mathcal{J}_\Omega) \geq \frac{1}{2}h$. Suppose in fact that this is not the case: $c_{\text{lin}}(\mathcal{J}_\Omega) < \frac{1}{2}h$. Then, by definition of c_{lin} , there does not exist any canonical transformation (affine, or not) sending the ball $B(\sqrt{\hbar})$ inside \mathcal{J}_Ω . But then there can be no affine symplectic transformation sending $B(\sqrt{\hbar})$

inside Ω , because of the 2-homogeneity property, and this contradicts the assumption $c_{\text{lin}}(\Omega) \geq \frac{1}{2}h$. Since \mathcal{J}_Ω is an ellipsoid, we can find a positive-definite $6N \times 6N$ matrix Σ such that \mathcal{J}_Ω consists of all phase space points $z = (x, p)^T$ satisfying the condition

$$\frac{1}{2}z^T \Sigma^{-1} z \leq 1. \quad (21)$$

The notation suggests that Σ can be viewed as a statistical covariance matrix, so let us write it in the block-matrix form

$$\Sigma = \begin{pmatrix} \Sigma_{XX} & \Sigma_{XP} \\ \Sigma_{PX} & \Sigma_{PP} \end{pmatrix}$$

where the blocks Σ_{XX} , $\Sigma_{XP} = \Sigma_{PX}^T$, and Σ_{PP} are $3N \times 3N$ matrices, which we find pleasant to write as $\Sigma_{XX} = (\text{Cov}(X_j, X_k))_{j,k}$, $\Sigma_{XP} = \Sigma_{PX}^T = (\text{Cov}(X_j, P_k))_{j,k}$, and $\Sigma_{PP} = (\text{Cov}(P_j, P_k))_{j,k}$. It is not difficult to prove (see de Gosson [10, 11]) that the equation (21) characterizing the John ellipsoid is rigorously equivalent to the set of inequalities

$$(\Delta X_j)^2 (\Delta P_j)^2 \geq (\text{Cov}(X_j, P_j))^2 + \frac{1}{4}\hbar^2 \quad (22)$$

where $(\Delta X_j)^2 = \text{Cov}(X_j, X_j)$ and $(\Delta P_j)^2 = \text{Cov}(P_j, P_j)$. The observant reader will recognize here the strong form of the Heisenberg uncertainty principle, due to Robertson [25] and Schrödinger⁶ [27]; it implies of course at once the textbook inequalities $\Delta X_j \Delta P_j \geq \frac{1}{2}\hbar$ if one neglects the covariances.

The reader who has had the patience to follow me so far certainly thinks that I have been tricking him somewhere. This is not the case; as we have shown in [10, 11] the inequalities (22) are mathematically equivalent to the statement that $c(\mathcal{J}_\Omega) \geq \frac{1}{2}h$ for every symplectic capacity c ; this statement is in turn equivalent to the matrix condition

$$\Sigma + \frac{i\hbar}{2}J \text{ is semi-definite positive} \quad (23)$$

well-known from quantum optics (see for instance [23, 29, 30]; we have reviewed the result in [11]). The proof of the equivalence between (21) and (22) simply relies on elementary linear algebra, using formula (20) which also applies to the linear symplectic capacity.

It turns out –and *this* is the important point!– that the inequalities (22) are conserved in time under Hamiltonian evolution. Thus, if condition (22)

⁶Angelow and Batoni have translated Schrödinger's paper in English in [1].

is true at some initial moment, then it will be true in the future, and was true in the past. Let us show this in the case of linear (or affine) flows. Returning to the convex phase space region Ω considered previously we assume again that $c_{\text{lin}}(\Omega) \geq \frac{1}{2}h$. After time t this region will be a new convex set Ω_t with same symplectic capacity (because c_{lin} is invariant under symplectic affine flows). We thus have $c_{\text{lin}}(\Omega_t) \geq \frac{1}{2}h$. It now suffices to consider the John ellipsoid \mathcal{J}_{Ω_t} , and to introduce the corresponding covariance matrix

$$\Sigma_t = \begin{pmatrix} \Sigma_{XX,t} & \Sigma_{XP,t} \\ \Sigma_{PX,t} & \Sigma_{PP,t} \end{pmatrix}$$

leading to the uncertainty inequalities (22) at time t .

The generalization to arbitrary Hamiltonian flows goes along the same lines, but is a little bit harder. The main observation is that a generic Hamiltonian flow does not preserve the convexity and one can thus not in general associate to $f_t(\Omega)$ a John ellipsoid; however this difficulty can be bypassed by observing that the convex hull of $f_t(\Omega)$ indeed contains such an ellipsoid. We will give a detailed study of the general case in a forthcoming work.

The reader should perhaps not be too surprised by the emergence of the uncertainty principle from classical considerations. It seems today sufficiently well-known that the uncertainty principle is really not enough for characterizing a quantum state (except in the Gaussian case). In a recent paper de Gosson and Luef [13] have discussed this fact from a mathematical point of view; our reflections were inspired by a previous paper by Man'ko et al. [22].

3.4 A topological reformulation of the uncertainty principle?

These *mathematical* facts tend to show –to paraphrase what Basil Hiley wrote in the foreword to my book [8]– that it is as if “... the uncertainty principle has left a ”footprint” in classical mechanics, and conversely”. They suggest that, perhaps, the most general formulation of the uncertainty of quantum mechanics could be topological. For instance one could envisage that phase space is coarse-grained, not by cubic cells with volume h^{3N} as is customary in statistical mechanics, but rather by arbitrary cells \mathcal{B} with symplectic capacity $c(\mathcal{B}) = \frac{1}{2}h$ (for some, or maybe every, symplectic capacity c). I have called such cells “quantum blobs” elsewhere; in [9] I actually showed that the consideration of quantum blobs as the finest possible coarse-graining can be applied to all quantum systems with completely integrable classical counterpart to recover the ground level energy. My attempts to

use these quantum blobs to also get the excited states have failed until now. Probably some refinement of Gromov’s non-squeezing theorem might be needed. Perhaps, symplectic packing techniques as exposed in Schlenk’s book [26]. Another very appealing possibility would to use techniques from a generalization of symplectic geometry, known as contact geometry. That this approach might be promising is clear from the paper [5] by Elisahberg et al. where my consideration of “small ellipsoids” is taken up from this point of view.

4 Concluding Remarks

With some afterthoughts the facts which we have exposed are not so surprising, after all. One should not forget that quantum mechanics (at least in its Schrödinger formulation) is built from the very beginning on classical (Hamiltonian) mechanics. The operator $H(x, -i\hbar\nabla_x)$ appearing in the Schrödinger equation is not pulled out of thin air; it is obtained using some “quantization rule” from a very classical object, namely the Hamiltonian function. Quantum mechanics appears from this viewpoint as a refinement of Hamiltonian mechanics; to support this claim (already made by Mackey [21] some years ago, although in a different context) it suffices that the variant of quantum mechanics known as *deformation quantization*.

Of course, these facts do not mean that there is no such thing as “true” quantum mechanics! Planck’s constant plays, as a physical constant, already a primordial role in the understanding of what a mixed state is. There is a very interesting notion, that of Wigner spectrum, due to Narcowich (see [23] and the references therein). The Wigner spectrum allows to characterize those self-adjoint operators $\hat{\rho}$ with trace one which really are mixed quantum states (a number ε is in the Wigner spectrum of $\hat{\rho}$ if, when one replaces \hbar by ε , the operator $\hat{\rho}$ remains semipositive-definite). It has recently been shown by Dias and Prata [4] that the only pure states having full Wigner spectrum $[0, \hbar]$ are Gaussian states. This fact, which is not a posteriori so surprising because Gaussians are the quantum equivalent of phase space points, indicates that in general one cannot let vary Planck’s constant without risking inconsistencies.

Perhaps all this could be understood from the standpoint exposed in Bohm and Hiley [3]; the ideas of “implicate order” exposed there are philosophically quite appealing. I will not discuss such a perspective here, if only because of lack of competence.

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